

BUCHSBAUM STANLEY–REISNER RINGS WITH MINIMAL MULTIPLICITY

NAOKI TERAJ AND KEN-ICHI YOSHIDA

ABSTRACT. In this paper, we study non-Cohen–Macaulay Buchsbaum Stanley–Reisner rings with linear free resolution. In particular, for given integers c, d, q with $c \geq 1, 2 \leq q \leq d$, we give an upper bound $h_{c,d,q}$ on the dimension of the unique non-vanishing homology $\tilde{H}_{q-2}(\Delta; k)$ of a d -dimensional Buchsbaum ring $k[\Delta]$ with q -linear resolution and codimension c . Also, we discuss about existence for such Buchsbaum rings with $\dim_k \tilde{H}_{q-2}(\Delta; k) = h$ for any h with $0 \leq h \leq h_{c,d,q}$, and prove an existence theorem in the case of $q = d = 3$ using the notion of Cohen–Macaulay linear cover. On the other hand, we introduce the notion of Buchsbaum Stanley–Reisner rings with minimal multiplicity of type q , which extends the notion of Buchsbaum rings with minimal multiplicity defined by Goto. As an application, we give many examples of Buchsbaum Stanley–Reisner rings with q -linear resolution.

INTRODUCTION

For any simplicial complex Δ on $V = \{x_1, \dots, x_n\}$, the homogeneous reduced k -algebra $k[\Delta] = k[X_1, \dots, X_n]/I_\Delta$, where I_Δ is the ideal generated by all square-free monomials $X_{i_1} \cdots X_{i_p}$ such that $\{x_{i_1}, \dots, x_{i_p}\} \notin \Delta$, is called the *Stanley–Reisner ring* of Δ . In recent years, the class of Stanley–Reisner rings is one of important classes in the theory of commutative algebra.

In [EiGo], Eisenbud and Goto investigated rings with linear resolution and showed the significance of this property. Then from many viewpoints, they have widely studied. Let us pick up some important results in the class of Stanley–Reisner rings. Fröberg [Fr1, Fr2] classified all Δ for which $k[\Delta]$ has 2-linear resolution. Hibi [Hi2] gave a necessary and sufficient condition for a Buchsbaum Stanley–Reisner ring to have linear resolution in terms of the reduced homology of the simplicial complex and the a -invariants of its links.

Also, there is a well-known criterion for a Cohen–Macaulay (Stanley–Reisner) ring to have linear resolution in terms of its h -vector or its multiplicity with given initial degree and codimension (see e.g. [EiGo]). However, as for Buchsbaum case, it seems that there is no such a criterion. Hence, in this paper, we investigate the structure of Buchsbaum Stanley–Reisner rings with linear resolution in connection with their multiplicities.

The purpose of this paper is divided into the following three pieces:

- (I): To give fundamental properties of Buchsbaum Stanley–Reisner rings with linear resolution.

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- (II): To introduce the notion of Buchsbaum Stanley–Reisner rings with minimal multiplicity of type q for any integer $q \geq 2$.
- (III): To construct 3-dimensional Buchsbaum Stanley–Reisner rings having 3-linear resolution with given parameters (codimension, dimension of the first reduced homology).

Let us explain the organization of this paper.

After recalling the notation and the terminology, in Section 2, we give fundamental properties of Buchsbaum Stanley–Reisner rings with linear resolution. Now let $A = k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution, and put $\text{codim } A = c (\geq 1)$. Then $q \leq d + 1$. If $q = d + 1$, then Δ is a $(d - 1)$ -skeleton of 2^V (Proposition 2.2). So we may assume that $2 \leq q \leq d$. Then $H_m^i(A) = 0$ for all $i \neq q - 1, d$; $H_m^{q-1}(A) \cong \tilde{H}_{q-2}(\Delta; k)$. Thus it seems that $h := \dim_k H_m^{q-1}(A)$ is an important invariant of A . From this point of view, we determined the h -vector of A and proved an inequality:

$$0 \leq h \leq h_{c,d,q} := \frac{(c + q - 2) \cdots (c + 1)c}{d(d - 1) \cdots (d - q + 2)};$$

see Theorem 2.6. Also, we pose the following conjecture:

Conjecture 2.9. Let d, c, q, h be integers with $c \geq 1, h \geq 0$, and $2 \leq q \leq d$. Then the following conditions are equivalent:

- (1) There exists a Buchsbaum Stanley–Reisner ring $A = k[\Delta]$ with q -linear resolution such that $\dim A = d$, $\text{codim } A = c$ and $\dim_k H_m^{q-1}(A) = h$.
- (2) The above inequality $0 \leq h \leq h_{c,d,q}$ holds.

The coarse version, which we solve in Section 4, of this conjecture is given by Hibi in [Hi2].

Hibi’s problem. Construct a Buchsbaum complex of dimension $d - 1$ with q -linear resolution for any given integers q, d with $2 \leq q \leq d$.

In Section 3, we study the Alexander dual of Buchsbaum complexes with linear resolution. In fact, we prove that $k[\Delta]$ is Buchsbaum with q -linear resolution if and only if $k[\Delta^*]$ (Δ^* denotes the Alexander dual of Δ) has almost c -linear resolution and $\beta_{qj}(k[\Delta^*]) = 0$ for all $j \neq c + q - 1, c + d$; see Theorem 3.2.

In Section 4, we introduce the notion of minimal multiplicity of type q for Buchsbaum Stanley–Reisner rings, and investigate its property.

In [Go2], Goto defined Buchsbaum local rings with minimal multiplicity, and he proved that they have 2-linear resolutions; see [Go1, Go2]. We generalize this notion in the class of Stanley–Reisner rings (Proposition 4.2). Namely, we prove the following theorem, which is a main result in this paper.

Theorem 4.3. Let $A = k[\Delta]$ be a Buchsbaum Stanley–Reisner ring with $\text{codim } A = c$ and $\text{indeg } A = q$. Then

- (1) The following inequality holds:

$$e(A) \geq \frac{c + d}{d} \binom{c + q - 2}{q - 2}.$$

- (2) If the equality holds in (1), then it has q -linear resolution.

We say that A has *minimal multiplicity of type q* if one of the following equivalent conditions (see Theorem 4.5 for more details):

- (1) The equality holds in Theorem 4.3(1).
- (2) A has q -linear resolution and $\dim_k H_m^{q-1}(A) = h_{c,d,q}$.
- (3) $a(A) = q - d - 2$.
- (4) $k[\Delta^*]$ is Cohen–Macaulay with pure and almost linear resolution with a -invariant 0.

In the proof of Theorem 4.3, *Hochster’s formula* and *Ho–Miyazaki theorem* play important roles. Also, using the same idea of the proof of Theorem 4.3, we can prove a generalization of *Hibi’s criterion*, which asserts that $A = k[\Delta]$ has q -linear resolution if and only if $\tilde{H}_{q-1}(\Delta; k) = 0$ and $a(k[\text{link}_\Delta \{x_i\}]) \leq q - d$ for all $i = 1, \dots, n$; see Theorem 4.4.

On the other hand, as an application of Theorems 3.2 and 4.5, we prove that the Alexander dual of the boundary complex of a cyclic polytope is Buchsbaum with minimal multiplicity in our sense. In particular, this example gives an affirmative answer to the above Hibi’s problem. See Section 4 for more examples.

In Section 5, we introduce the notion of Cohen–Macaulay cover, and prove that Conjecture 2.9 is true for $q = d = 3$. Let Δ be a pure simplicial complex on vertex set V with $\text{indeg } k[\Delta] = \dim k[\Delta] = d$. Then $\tilde{\Delta}$ is said to be a *Cohen–Macaulay cover* of Δ over a field k if it is a $(d-1)$ -dimensional simplicial complex on the same vertex set V as containing Δ and $k[\tilde{\Delta}]$ is Cohen–Macaulay with d -linear resolution. The notion of Cohen–Macaulay cover is very useful to attack the above conjecture.

Now consider the case of $q = d$ in the above Conjecture 2.9. Let c, d and h be integers with $c \geq 1$, $d \geq 2$ and $0 \leq h \leq h_{c,d,d} = \frac{(c+d-2) \cdots (c+1)c}{d!}$. Also, let Δ^{\min} be a $(d-1)$ -dimensional Buchsbaum simplicial complex with d -linear resolution and $\dim_k H_m^{d-1}(k[\Delta^{\min}]) = \lfloor h_{c,d,d} \rfloor$. Then we can prove the following:

Corollary 5.3. If such a complex Δ^{\min} exists, then there exists a Cohen–Macaulay cover $\tilde{\Delta}$ of Δ^{\min} .

Theorem 5.7. Let $\Delta^- \subseteq \Delta \subseteq \Delta^+$ be simplicial complexes on $V = \{x_1, \dots, x_n\}$. If both $k[\Delta^-]$ and $k[\Delta^+]$ are Buchsbaum Stanley–Reisner rings with d -linear resolutions, then so is $k[\Delta]$.

Thus we can reduce Conjecture 2.9 to the existence of Δ^{\min} . For $d = 3$, we show that Hanano’s example is turned out to be Δ^{\min} (Example 4.10). Combining the above result with Corollary 2.12 and Example 4.7, we have:

Theorem. Conjecture 2.9 is true if either one of the following conditions are satisfied:

- (1) $d \leq 3$.
- (2) $q = 2$.
- (3) $h_{c,d,q} = 1$.

1. PRELIMINARIES

We first fix notation. Let \mathbb{N} (resp. \mathbb{Z}) denote the set of nonnegative integers (resp. integers). Put $[n] = \{1, 2, \dots, n\}$ for any positive integer n . Let $\#(W)$ denote the cardinality of a set W .

We recall some notation on simplicial complexes and Stanley–Reisner rings according to [St]. We refer the reader to e.g. [BrHe], [Hi1], [Hoc] and [St] for the detailed information about combinatorial and algebraic background.

1.1 (Simplicial complex, face). A *simplicial complex* Δ on the *vertex set* $V = \{x_1, \dots, x_n\}$ is a collection of subsets of V such that

- (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq n$;
- (ii) $F \in \Delta, G \subseteq F \implies G \in \Delta$.

Each element F of Δ is called a *face* of Δ . A face F is called an *i-face* if $\#(F) = i+1$. We set $d = \max\{\#(F) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d-1$. A maximal face is called a *facet*. We say that Δ is *pure* if every facet has the same cardinality.

Given a subset W of V , the *restriction* of Δ is the subcomplex

$$\Delta_W = \{F \in \Delta \mid F \subseteq W\}.$$

On the other hand, if F is a face of Δ , then we define the subcomplex $\text{link}_\Delta(F)$, which is called the *link of F* as follows:

$$\text{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

In particular, $\text{link}_\Delta(\emptyset) = \Delta$.

In the following, let Δ be a $(d-1)$ -dimensional simplicial complex on $V = \{x_1, \dots, x_n\}$. Let $S = k[x_1, \dots, x_n]$ be a polynomial ring in n -variables over a field k . Here, we identify each $x_i \in V$ with the indeterminate x_i of S . Also, let $\mathfrak{m}_S = (x_1, \dots, x_n)S$ denote the homogeneous maximal ideal of S .

1.2 (*h*-vector, Reduced homology). Let $f_i = f_i(\Delta)$, $0 \leq i \leq d-1$, denote the numbers of i -faces in Δ . We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ the *f-vector* of Δ . Also, we define the *h-vector* $h(\Delta) = (h_0, h_1, \dots, h_d)$ of Δ by

$$(1.1) \quad \sum_{i=0}^d \frac{f_{i-1}t^i}{(1-t)^i} = \frac{h_0 + h_1t + \dots + h_d t^d}{(1-t)^d}.$$

Let $C_p(\Delta)$ be the k -vector space generated by all p -faces of Δ and we define the differential map $\partial_{p+1}: C_{p+1}(\Delta) \rightarrow C_p(\Delta)$ as follows:

$$\partial_{p+1}(\{x_{i_1}, \dots, x_{i_{p+1}}\}) = \sum_{j=1}^{p+1} (-1)^j \{x_{i_1}, \dots, \widehat{x_{i_j}}, \dots, x_{i_{p+1}}\}.$$

Then $C_\bullet(\Delta)$ is a bounded complex and $\tilde{H}_p(\Delta; k) = H_p(C_\bullet(\Delta))$ is called the *ith reduced simplicial homology group* of Δ with values in k . Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_p(\{\emptyset\}; k) = \begin{cases} 0 & (p \geq 0); \\ k & (p = -1). \end{cases}$$

Also, note that $\tilde{H}_p(\Delta; k) = 0$ if $p \geq d = \dim \Delta + 1$ or $p \leq -2$.

1.3 (Stanley–Reisner ring). Let I_Δ be the ideal of S which is generated by square-free monomials $x_{i_1} \cdots x_{i_r}$, $1 \leq i_1 < \dots < i_r \leq n$, with $\{x_{i_1}, \dots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := S/I_\Delta$ is the *Stanley–Reisner ring* of Δ over k . We consider $k[\Delta]$ as a graded algebra $k[\Delta] = \bigoplus_{j \geq 0} k[\Delta]_j$ with the standard grading, i.e., each $\deg x_i = 1$.

Let $e(k[\Delta])$ denote the *multiplicity* of $k[\Delta]$. It is well known that $e(k[\Delta]) = f_{d-1}(\Delta) = \sum_{i=0}^d h_i(\Delta)$.

Theorem 1.4 (Hochster's formula on the local cohomology modules). *Let $k[\Delta]$ be a Stanley–Reisner ring and \mathfrak{m} the unique homogeneous maximal ideal of $k[\Delta]$. Then the Hilbert series of the i th local cohomology module $H_{\mathfrak{m}}^i(k[\Delta])$ is given by*

$$(1.2) \quad F(H_{\mathfrak{m}}^i(k[\Delta]), t) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-\#(F)-1}(\text{link}_{\Delta} F; k) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{\#(F)},$$

where $F(M, t) = \sum_{n \in \mathbb{Z}} \dim_k M_n t^n$ for a graded S -module M with $\dim_k M_n < \infty$ for all $n \in \mathbb{Z}$.

In particular, we have

- (1) $[H_{\mathfrak{m}}^i(k[\Delta])]_j = 0$ for all $j \geq 1$.
- (2) $[H_{\mathfrak{m}}^i(k[\Delta])]_0 \cong \tilde{H}_{i-1}(\Delta; k)$.
- (3) For all $p \geq 1$,

$$\dim_k [H_{\mathfrak{m}}^i(k[\Delta])]_{-p} = \sum_{F \in \Delta, 1 \leq \#(F) \leq p} \dim_k \tilde{H}_{i-\#(F)-1}(\text{link}_{\Delta} F; k).$$

Also, if we define the a -invariant of a d -dimensional graded ring A (see [GoWa]) by

$$a(A) = \max\{m \in \mathbb{Z} \mid [H_{\mathfrak{m}}^d(A)]_m \neq 0\},$$

then we always have $a(k[\Delta]) \leq 0$.

1.5 (Cohen–Macaulay, Buchsbaum complex). Δ is called *Cohen–Macaulay* over k if it satisfies one of the following equivalent conditions:

- (1) $k[\Delta]$ is Cohen–Macaulay.
- (2) $H_{\mathfrak{m}}^i(k[\Delta]) = 0$ for all $i < d$. That is, $\text{depth } k[\Delta] = \dim k[\Delta]$.
- (3) $\tilde{H}_i(\text{link}_{\Delta}(F); k) = 0$ for every $F \in \Delta$ and for every $i < \dim \text{link}_{\Delta}(F)$.

Also, Δ is called *Buchsbaum* over k if it satisfies one of the following equivalent conditions:

- (1) $k[\Delta]$ is Buchsbaum. That is, $l(k[\Delta]/J) - e(J)$ (this invariant is written as $I(k[\Delta])$) is independent on the choice of homogeneous parameter ideal J of $k[\Delta]$, where $e(J)$ denotes the multiplicity of J .
- (2) $k[\Delta]$ is *(F.L.C.)*, i.e., $l(H_{\mathfrak{m}}^i(k[\Delta])) < \infty$ for all $i < d$.
- (3) $H_{\mathfrak{m}}^i(k[\Delta]) = [H_{\mathfrak{m}}^i(k[\Delta])]_0 = \tilde{H}_{i-1}(\Delta; k)$ for all $i < d$.
- (4) Δ is pure and $\tilde{H}_i(\text{link}_{\Delta}(F); k) = 0$ for every $F(\neq \emptyset) \in \Delta$ and for every $i < \dim \text{link}_{\Delta}(F)$.

When this is the case,

$$I(k[\Delta]) = \sum_{i=0}^{d-1} \binom{d-1}{i} l(H_{\mathfrak{m}}^i(k[\Delta])) = \sum_{i=0}^{d-1} \binom{d-1}{i} \dim_k \tilde{H}_{i-1}(\Delta; k).$$

Note that $k[\Delta]$ is Cohen–Macaulay if and only if it is Buchsbaum and $\tilde{H}_i(\Delta; k) = 0$ for all $i < d-1$.

1.6 (Regularity, Linear resolution). Let $A = k[A_1] = S/I$ be a homogeneous k -algebra.

A graded *minimal free resolution* (abbr., MFR) of A over S is an exact sequence

$$0 \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{p,j}(A)} \xrightarrow{\varphi_p} \dots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}(A)} \xrightarrow{\varphi_1} S \rightarrow A \rightarrow 0,$$

where $S(j)$, $j \in \mathbb{Z}$, denotes the graded module $S(j) = \bigoplus_{n \in \mathbb{Z}} S_{j+n}$ and “minimal” means that $\varphi_i \otimes_A A/\mathfrak{m} = 0$ for all i . We say that $\beta_i(A) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(A)$ the i th *Betti number* of A over S .

The *Castelnuovo–Mumford regularity* is defined by

$$\text{reg } A := \max\{j - i \mid \beta_{i,j}(A) \neq 0\}.$$

Also, the *initial degree* of A is defined by

$$\text{indeg } A := \min\{j \mid I_j \neq 0\} = \min\{j \mid \beta_{1,j}(A) \neq 0\}.$$

Notice the following fact:

- (1) $\beta_{i,j}(A) = \dim_k \text{Tor}_S^i(A, S/\mathfrak{m}_S)_j$.
- (2) $p = n - \text{depth } A$ by Auslander–Buchsbaum formula.
- (3) $\beta_{i,j}(A) = 0$ for all $j < i + \text{indeg } A - 1$.
- (4) $\text{reg } A \geq \text{indeg } A - 1$.
- (5) $\text{reg } A = \inf\{r \in \mathbb{Z} \mid [H_{\mathfrak{m}}^i(A)]_j = 0 \text{ for all } j > r - i\}$; see [EiGo].

A has *q-linear resolution* (abbr., A is q -linear or Δ is q -linear over k) if $\text{reg } A = \text{indeg } A - 1 = q - 1$ (e.g., [Oo]), that is, its graded minimal free resolution of A is written as the following form:

$$0 \rightarrow S(-(q+p-1))^{\beta_p} \rightarrow S(-(q+p-2))^{\beta_{p-1}} \rightarrow \dots \rightarrow S(-q)^{\beta_1} \rightarrow S \rightarrow A \rightarrow 0.$$

In the case of Buchsbaum homogeneous k -algebras, the following criterion for having a q -linear resolution is known. See also Theorems 2.3 and 4.4.

Theorem 1.7 (cf. [EiGo, Corollary 1.5]). *Let $A = k[A_1]$ be a homogeneous Buchsbaum k -algebra with $\text{indeg } A \geq q$ and $\dim A = d$. Put $\mathfrak{m} = A_+$, the unique homogeneous maximal ideal of A . Suppose that k is infinite. Then the following conditions are equivalent:*

- (1) A has q -linear resolution.
- (2) There exists a homogeneous system of parameters f_1, \dots, f_d such that $\mathfrak{m}^q = (f_1, \dots, f_d)\mathfrak{m}^{q-1}$.
- (3) $[H_{\mathfrak{m}}^i(A)]_j = 0$ for all $i \neq d$, $j \neq q - 1 - i$ and $[H_{\mathfrak{m}}^d(A)]_j = 0$ for all $j > q - d - 1$.

The Betti numbers of Stanley–Reisner rings can be calculated by the following formula.

Theorem 1.8 (Hochster’s formula on the Betti numbers [Hoc, Theorem 5.1]). *Let $k[\Delta]$ be the Stanley–Reisner ring of Δ over k . Then*

$$(1.3) \quad \beta_{i,j}(k[\Delta]) = \sum_{\substack{W \subseteq V \\ \#(W)=j}} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Also, we frequently use the following theorem; see [HoMi, Corollary 2.8].

Theorem 1.9 (Hoa–Miyazaki theorem). *Let $A = k[A_1]$ be a d -dimensional homogeneous Buchsbaum k -algebra. Then*

$$\text{reg } A \leq a(A) + d + 1.$$

2. BUCHSBAUM STANLEY–REISNER RINGS WITH q -LINEAR RESOLUTION

In this section, let us gather several properties of Buchsbaum Stanley–Reisner rings having a q -linear resolution.

Throughout this section, let Δ be a simplicial complex on $V = \{x_1, \dots, x_n\}$, and let $k[\Delta]$ denote the Stanley–Reisner ring of Δ over a field k , and \mathfrak{m} its homogeneous maximal ideal. Also, let c, d, q be given integers with $c \geq 1$ and $q, d \geq 2$.

Let us start giving fundamental results on Buchsbaum Stanley–Reisner rings with linear resolution.

Proposition 2.1. (cf. [EiGo]) *Suppose that $A = k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution, and put $\text{codim } A = c$. Then*

- (1) $q \leq d + 1$.
- (2) $H_{\mathfrak{m}}^i(A) = [H_{\mathfrak{m}}^i(A)]_0 \cong \tilde{H}_{i-1}(\Delta; k) = 0$ for all $i \neq q - 1, d$.
- (3) $\text{reg } A = q - 1$ and $\text{indeg } A = q$.
- (4) $a(A) = q - d - 2$ or $q - d - 1$.

Also, suppose that $q \leq d$. Then

- (5) $\tilde{H}_{d-1}(\Delta; k) = 0$.
- (6) $\dim_k H_{\mathfrak{m}}^{q-1}(A) = \dim_k \tilde{H}_{q-2}(\Delta; k) = \beta_{c+d-q+1}$.

Proof. (3) is clear by definition. By Hochster’s formula, we have $\text{reg } A \leq d$ in general. Thus we get (1). Also, (2) easily follows from Theorem 1.7. Similarly we have $a(A) \leq q - d - 1$. On the other hand, $q - 1 = \text{reg } A \leq a(A) + d + 1$ by Hoa–Miyazaki Theorem. This implies that $a(A) \geq q - d - 2$. Hence we get (4).

Now, suppose that $q \leq d$. Since $\text{reg } A = d > q - 1$, we have that $\tilde{H}_{d-1}(\Delta; k) = [H_{\mathfrak{m}}^d(A)]_0 = 0$. Also, (6) follows from Theorem 1.8 if we put $j = c + d = n$ and $i = c + d - q + 1$. \square

In the following, we always assume that $q \leq d$. In fact, we can completely characterize Stanley–Reisner rings with $(d + 1)$ -linear resolution as follows:

Proposition 2.2. *Let $A = k[\Delta]$ be a d -dimensional Stanley–Reisner ring on V . Then the following conditions are equivalent:*

- (1) A has $(d + 1)$ -linear resolution.
- (2) $\text{indeg } A = d + 1$.
- (3) $I_{\Delta} = (x_{i_1} \cdots x_{i_{d+1}} \mid 1 \leq i_1 < \cdots < i_d < i_{d+1} \leq n)$. That is, Δ is the $(d - 1)$ -skeleton of the standard n -simplex 2^V .

When this is the case, A is Cohen–Macaulay.

Proof. (1) \implies (2) is trivial. Conversely, suppose (2). Then $d \geq \text{reg } A \geq \text{indeg } A - 1 = d$. Thus A has $(d + 1)$ -linear resolution.

(3) \implies (2) is trivial. Conversely, suppose that $\text{indeg } A = d + 1$. Then I_{Δ} does not contain any square-free monomial M with $\deg M \leq d$. Now suppose that some square-free monomial $x_{i_1} \cdots x_{i_{d+1}}$ of degree $(d + 1)$ is not contained in I_{Δ} . Then $\{x_{i_1}, \dots, x_{i_{d+1}}\} \in \Delta$. This contradicts the assumption that $\dim \Delta = d - 1$. Hence Δ is a $(d - 1)$ -skeleton of 2^V . Since 2^V is Cohen–Macaulay, so is Δ ; see e.g., [BrHe, Ex 5.1.23]. \square

The following theorem gives a criterion for having a q -linear resolution. In Section 4, we will improve this theorem.

Theorem 2.3 (Hibi's criterion [Hi2, Theorem (1.6)]). *Let $A = k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring. Put $\text{indeg } A = q \leq d$. Then A has q -linear resolution if and only if the following conditions are satisfied:*

- (i) $\tilde{H}_i(\Delta; k) = 0$ for all $i \neq q - 2$.
- (ii) $a(k[\text{link}_\Delta(\{x_i\})]) \leq q - d$ for all $i = 1, \dots, n$.

Now suppose that $A = k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution. Then $h := \dim_k \tilde{H}_{q-2}(\Delta; k) = \dim_k H_{\mathfrak{m}}^{q-1}(A)$ is an important invariant of Δ . From now on, we focus this invariant.

The following proposition may be known, but we give a proof for the readers' convenience.

Proposition 2.4. *Suppose that $A = k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution, and put $\text{codim } A = c$. If we put $h := \dim_k H_{\mathfrak{m}}^{q-1}(A)$, then the multiplicity $e(A)$ and the I -invariant $I(A)$ are given by*

$$(2.1) \quad e(A) = \binom{c+q-1}{q-1} - h \binom{d-1}{q-1} \quad \text{and} \quad I(A) = h \binom{d-1}{q-1}.$$

To prove the above proposition, we first show the following lemma. See also [EiGo].

Lemma 2.5. *Let $A = k[A_1]$ be a d -dimensional homogeneous Buchsbaum k -algebra with $\text{codim } A = c$ and $\text{indeg } A \geq q$. Then*

- (1) $e(A) \geq \binom{c+q-1}{q-1} - I(A)$.
- (2) Equality holds in (1) if and only if A has q -linear resolution.

Proof. We may assume that k is infinite.

(1) Let J be a homogeneous minimal reduction of \mathfrak{m} , that is, J is a homogeneous parameter ideal of A and $\mathfrak{m}^{r+1} = J\mathfrak{m}^r$ holds for some integer $r \geq 0$. Then $e(A) = e(J) = l_A(A/J) - I(A)$ since A is Buchsbaum. Also, since $B := A/J$ is a homogeneous Artinian k -algebra with $\dim_k B_1 = c$ and $\text{indeg } B \geq q$, we have

$$e(A) = l_A(A/J) - I(A) \geq l_A(A/J + \mathfrak{m}^q) - I(A) = \binom{c+q-1}{q-1} - I(A).$$

(2) Equality holds in (1) if and only if $\mathfrak{m}^q \subseteq J$. Since A is homogeneous, this yields that $\mathfrak{m}^q = J \cap \mathfrak{m}^q = J\mathfrak{m}^{q-1}$. Thus the statement follows from Theorem 1.7. \square

Proof of Proposition 2.4. Since $A = k[\Delta]$ has q -linear resolution, $H_{\mathfrak{m}}^i(A) = 0$ for all $i \neq q - 1, d$ and thus $I(A) = h \binom{d-1}{q-1}$. On the other hand, by Lemma 2.5, we have

$$e(A) = \binom{c+q-1}{q-1} - I(A) = \binom{c+q-1}{q-1} - h \binom{d-1}{q-1},$$

as required. \square

The following theorem plays an important role in this article.

Theorem 2.6. Suppose that $A = k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution ($q \leq d$), and put $c = \text{codim } A$ and $h = \dim_k H_{\mathfrak{m}}^{q-1}(A)$. Then the h -vector $(h_0, h_1, \dots, h_{q-1}, h_q, h_{q+1}, \dots, h_d)$ of Δ is

$$(2.2) \quad \left(1, c, \dots, \binom{c+q-2}{q-1}, -\binom{d}{q}h, \binom{d}{q+1}h, \dots, (-1)^{d-q+1}\binom{d}{d}h\right).$$

That is,

$$h_p = \binom{c+p-1}{p} \quad (1 \leq p \leq q-1), \quad h_p = (-1)^{p-q+1}\binom{d}{p}h \quad (q \leq p \leq d).$$

Also, h satisfies the following inequalities:

$$(2.3) \quad 0 \leq h \leq \frac{(c+q-2) \cdots (c+1)c}{d(d-1) \cdots (d-q+2)} =: h_{c,d,q}.$$

Proof. Since $\text{indeg } A = q$, one has

$$h_p = \binom{c+p-1}{p} \quad \text{for all } p = 0, 1, \dots, q-1.$$

On the other hand, by the similar argument as in the proof of [Te1, Theorem 2.1], we have

$$\begin{aligned} \dim_k[H_{\mathfrak{m}}^d(A)]_{-1} &= d \cdot h_d + h_{d-1} \\ \dim_k[H_{\mathfrak{m}}^d(A)]_{-2} &= \binom{d+1}{2}h_d + d \cdot h_{d-1} + h_{d-2} \\ &\dots\dots\dots \\ \dim_k[H_{\mathfrak{m}}^d(A)]_{-p} &= \binom{d+p-1}{p}h_d + \binom{d+p-2}{p-1}h_{d-1} + \dots + d \cdot h_{d-p+1} + h_{d-p} \\ &\dots\dots\dots \\ \dim_k[H_{\mathfrak{m}}^d(A)]_{q-d} &= \binom{2d-q-1}{d-q}h_d + \binom{2d-q-2}{d-q-1}h_{d-1} + \dots + d \cdot h_{q+1} + h_q. \end{aligned}$$

By Proposition 2.1, we have

$$(-1)^{d-1}h_d = \tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i \dim_k \tilde{H}_i(\Delta; k) = (-1)^q h$$

and $\dim_k[H_{\mathfrak{m}}^d(A)]_j = 0$ for all $j = -1, -2, \dots, q-d$. Solving the above equations, one can easily obtain that $h_p = (-1)^{p-q+1}\binom{d}{p}h$ for all $p = q, \dots, d-1, d$. Then

$$\binom{2d-q}{d-q+1}h_d + \dots + d \cdot h_q + h_{q-1} = \dim_k[H_{\mathfrak{m}}^d(A)]_{q-d-1} \geq 0$$

implies that

$$\binom{c+q-2}{q-1} - \binom{d}{q-1}h \geq 0.$$

Namely, $h \leq h_{c,d,q}$, as required. \square

Remark 2.7. Proposition 2.4 also follows from Theorem 2.6.

Remark 2.8. Under the same notation as in Theorem 2.6, let β_i be the i th Betti number of $k[\Delta]$ over S . Then the following identity holds:

$$(1-t)^c \left\{ \sum_{p=0}^{q-1} \binom{c+p-1}{p} t^p + \sum_{p=q}^d (-1)^{p-q+1} \binom{d}{p} h t^p \right\} = 1 - \sum_{i=1}^{n-q+1} \beta_i t^{q+i-1}.$$

Based on Theorem 2.6, we pose the following conjecture.

Conjecture 2.9. Let d, c, q, h be integers with $c \geq 1$, $h \geq 0$, and $2 \leq q \leq d$. Then the following conditions are equivalent:

- (1) There exists a Buchsbaum Stanley–Reisner ring $A = k[\Delta]$ with q -linear resolution such that $\dim A = d$, $\operatorname{codim} A = c$ and $\dim H_{\mathfrak{m}}^{q-1}(A) = h$.
- (2) The following inequality holds:

$$0 \leq h \leq h_{c,d,q} = \frac{(c+q-2) \cdots (c+1)c}{d(d-1) \cdots (d-q+2)}.$$

Remark 2.10 (Stanley). There is a $(d-1)$ -dimensional Cohen–Macaulay complex Δ with h -vector

$$(h_0, \dots, h_{q-1}) = \left(1, c, \binom{c+1}{2}, \dots, \binom{h+q-2}{q-1} \right),$$

which has q -linear resolution. See [BrHe, Theorem 5.1.15].

In particular, Conjecture 2.9 is true in the case of $h = 0$.

Now let $A = k[A_1]$ be a d -dimensional Buchsbaum homogeneous k -algebra. Let $e(A)$ (resp. $\operatorname{emb}(A) = \dim_k A_1$) denote the multiplicity (resp. the embedding dimension) of A . Then $\operatorname{emb}(A) \leq e(A) + \dim A + I(A) - 1$ holds in general, and A is said to have *maximal embedding dimension* if equality holds. Also, A has maximal embedding dimension if and only if it has 2-linear resolution or is isomorphic to a polynomial ring; see e.g. [Go1].

Fröberg ([Fr1, Fr2]) has determined the structure of Buchsbaum simplicial complexes with 2-linear resolution.

Proposition 2.11 (Fröberg [Fr2]). Let Δ be a $(d-1)$ -dimensional simplicial complex which is not a $(d-1)$ -simplex. Then the following conditions are equivalent:

- (1) $k[\Delta]$ is Buchsbaum with 2-linear resolution.
- (2) Δ is a finite disjoint union of $(d-1)$ -dimensional simplicial complexes Δ_i such that $k[\Delta_i]$ is Cohen–Macaulay of maximal embedding dimension.

Using the above proposition, we can show that the above conjecture is true for $q = 2$.

Corollary 2.12. Let d, c, h be integers with $d \geq 2$, $c \geq 1$, and $h \geq 0$. Then the following conditions are equivalent:

- (1) There exists a Buchsbaum Stanley–Reisner ring $A = k[\Delta]$ with 2-linear resolution such that $\dim A = d$, $\operatorname{codim} A = c$ and $\dim H_{\mathfrak{m}}^1(A) = h$.
- (2) The following inequality holds:

$$0 \leq h \leq h_{c,d,2} = \frac{c}{d}.$$

Proof. Note that there are many examples of Cohen–Macaulay Stanley–Reisner ring A over any field k of maximal embedding dimension such that $\text{codim } A = c$ and $\dim A = d$. Indeed, $k[X_0, \dots, X_c, Y_1, \dots, Y_{d-1}]/(X_i X_j \mid 0 \leq i < j \leq c)$ gives one of such examples.

To see (2) \implies (1), suppose that $c \geq dh$. We may assume that $h > 0$. Take any simplicial complex Δ_0 for which $k[\Delta_0]$ is Cohen–Macaulay of maximal embedding dimension and $\text{codim } k[\Delta_0] = c - dh$. Let Δ be a disjoint union of Δ_0 and $(d-1)$ -simplexes $\Delta_1, \dots, \Delta_h$. Then $A = k[\Delta]$ is Buchsbaum with 2-linear resolution by Proposition 2.11. Also, we have $\text{codim } A = (c - dh) + d \cdot h = c$, $\dim A = d$ and $\dim_k H_m^1(A) = h$. \square

On the other hand, in the case of $q \geq 3$, it seems to be difficult to construct examples of complexes having a q -linear resolution with given parameters. But, in Section 5, we will give an affirmative answer in the case of $q = d = 3$ using the notion of “Cohen–Macaulay cover”.

3. ALEXANDER DUALITY OF BUCHSBAUM COMPLEX WITH LINEAR RESOLUTION

In this section, we characterize the Alexander dual of Buchsbaum simplicial complexes with linear resolution. As an application, we give some examples of Buchsbaum complexes with linear resolution using cyclic polytopes. We first recall some basic results on Alexander duality of simplicial complexes.

3.1 (Alexander duality). Let Δ be a $(d-1)$ -dimensional simplicial complex on $V = \{x_1, \dots, x_n\}$. The *Alexander dual* of Δ is defined by

$$\Delta^* := \{F \subseteq V \mid V \setminus F \notin \Delta\}.$$

Suppose that $c := n - d \geq 2$ and $\text{indeg } A = q \geq 2$. Then Δ^* is a simplicial complex on V . Also, the following statements hold: see [EaRe, Te2] for details.

- (1) (Alexander Duality) $\tilde{H}_{i-2}(\Delta^*; k) \cong \tilde{H}^{n-i-1}(\Delta; k)$ for all i .
- (2) $(\Delta^*)^* = \Delta$.
- (3) $\dim k[\Delta^*] + \text{indeg } k[\Delta] = c + d = n$. In particular,
$$\dim k[\Delta^*] = c + d - q, \quad \text{indeg } k[\Delta^*] = c, \quad \text{and} \quad \text{codim } k[\Delta^*] = q.$$
- (4) The Betti numbers of $k[\Delta^*]$ are given by the formula

$$\beta_{i,j}(k[\Delta^*]) = \sum_{\substack{F \in \Delta \\ \#(F) = c + d - j}} \dim_k \tilde{H}_{i-2}(\text{link}_\Delta F; k).$$

- (5) $k[\Delta]$ has linear resolution if and only if $k[\Delta^*]$ is Cohen–Macaulay. In fact, we have

$$\text{reg } k[\Delta] - \text{indeg } k[\Delta] + 1 = \dim k[\Delta^*] - \text{depth } k[\Delta^*].$$

Described as above, if Δ is Cohen–Macaulay with linear resolution, then so is Δ^* . Thus it is a natural to ask

“What can we say about the Alexander dual of a *Buchsbaum* simplicial complex with linear resolution?”

An answer to this question is that “The Alexander dual of such a complex has *almost linear resolution* (see [EiGo]) with suitable conditions”. To be precise, we have:

Theorem 3.2. *Let c, d, q be integers with $c \geq 2$, $2 \leq q \leq d$. Let $A = k[\Delta]$ be a d -dimensional Stanley–Reisner ring with $\text{codim } A = c$ and $\text{indeg } A = q$, and let Δ^* denote the Alexander dual of Δ . Put $A^* := k[\Delta^*]$. Then the following conditions are equivalent:*

- (1) *A is Buchsbaum with q -linear resolution.*
- (2) *A^* is Cohen–Macaulay with almost c -linear resolution and the graded minimal free resolution of A^* over S can be written as follows:*

$$0 \rightarrow F_q \rightarrow F_{q-1} = S(-(c+q-2))^{\beta_{q-1}^*} \rightarrow \cdots \rightarrow F_1 = S(-c)^{\beta_1^*} \rightarrow S \rightarrow A^* \rightarrow 0,$$

$$\text{where } F_q = S(-(c+d))^{\beta^*} \oplus S(-(c+q-1))^{\beta^{*'}}.$$

When this is the case, $\beta^* = \dim_k H_{\mathfrak{m}}^{q-1}(A)$ and $\beta^{*'} = \dim_k H_{\mathfrak{m}}^d(A)_{q-d-1}$.

Proof. We may assume that A has q -linear resolution and A^* is Cohen–Macaulay by 3.1.

(1) \implies (2) : To see (2), we must show that $\beta_{ij}(k[\Delta^*]) = 0$ for all pairs (i, j) with $0 \leq i \leq q$ and $j > c+i-1$ except $(i, j) = (q, c+d)$.

Let F be a face of Δ with $\#(F) = c+d-j$. First suppose that $j = c+d$. Then $F = \emptyset$. If $i \leq q-1$, then $\tilde{H}_{i-2}(\Delta; k) = 0$ by Proposition 2.1. Next suppose that $c+i-1 < j < c+d$. Then $F \neq \emptyset$. Since $i-2 < j-c-1 = d-\#(F)-1 = \dim \text{link}_{\Delta} F$, the Buchsbaumness of Δ implies that $\tilde{H}_{i-2}(\text{link}_{\Delta} F; k) = 0$. Thus we get the required vanishing.

(2) \implies (1) : First, note that Δ is pure. Indeed, I_{Δ^*} is minimally generated by the elements $x_{j_1} \cdots x_{j_c}$ for which $V \setminus \{x_{j_1}, \dots, x_{j_c}\}$ is a maximal face of Δ .

To see the Buchsbaumness of $k[\Delta]$, let F be any non-empty face of Δ and let i be an integer with $i < d-\#(F)+1$. Put $j = c+d-\#(F)$. Then one can easily see that $\beta_{ij}(k[\Delta^*]) = 0$ by the assumption (2). This implies that $\tilde{H}_{i-2}(\text{link}_{\Delta}(F); k) = 0$. Hence $k[\Delta]$ is Buchsbaum, as required.

Putting $i = q$, $j = n (= c+d)$, we have

$$\beta^* = \beta_{q,n}(k[\Delta^*]) = \dim_k \tilde{H}_{q-2}(\Delta; k) = \dim_k H_{\mathfrak{m}}^{q-1}(A).$$

On the other hand, since $a(k[\Delta]) \leq q-d-1$, we have $\tilde{H}_{d-\#(F)-2}(\text{link}_{\Delta} F; k) = 0$ for every face F with $1 \leq \#(F) \leq d-q$. Thus by Hochster's formula we have

$$\begin{aligned} \dim_k [H_{\mathfrak{m}}^d(k[\Delta])]_{q-d-1} &= \sum_{F \in \Delta, \#(F)=d-q+1} \dim_k \tilde{H}_{q-2}(\text{link}_{\Delta} F; k) \\ &= \beta_{q, c+q-1}(k[\Delta^*]) = \beta^{*'} . \end{aligned}$$

□

Corollary 3.3. *Let c, q and d be integers with $c \geq 2$ and $2 \leq q \leq d$. Put $n = c+d$. Let Γ be a simplicial complex on V . Suppose that $k[\Gamma]$ is a $(n-q)$ -dimensional Gorenstein ring with almost c -linear resolution and $a(k[\Gamma]) = 0$. Let $\Delta = \Gamma^*$ be the Alexander dual of Γ . Then $k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution.*

Proof. Since $k[\Gamma]$ is Gorenstein and thus is Cohen–Macaulay, the length of the graded MFR of $k[\Gamma]$ is equal to $n - \dim k[\Gamma] = q$ by Auslander–Buchsbaum formula. Also, since the graded MFR of $k[\Gamma]$ is almost c -linear, it can be written as follows:

$$0 \rightarrow F_q \rightarrow F_{q-1} = S(-(c+q-2))^{\beta_{q-1}^*} \rightarrow \cdots \rightarrow F_1 = S(-c)^{\beta_1^*} \rightarrow S \rightarrow k[\Gamma] \rightarrow 0,$$

where $S = k[X_1, \dots, X_n]$ and $F_q = S(-\epsilon)$. Then $\epsilon - n = a(k[\Gamma]) = 0$; hence $\epsilon = n = c + d$. Hence $k[\Delta]$ has q -linear resolution by Theorem 3.2. \square

Let n, f be integers with $n \geq f + 1$. Consider the algebraic curve $M \subseteq \mathbb{R}^f$, defined by parametrically by $x(t) = (t, t^2, \dots, t^f)$, $t \in \mathbb{R}$. Let $C(n, f)$ be the convex hull of any distinct n -points over $M \subseteq \mathbb{R}^f$. Then $C(n, f)$ becomes a simplicial f -polytope. It is called a *cyclic polytope* with n vertices;

It is well-known that any simplicial f -polytope P with n vertices satisfies $0 \leq h_i(P) \leq \binom{n-f+i-1}{i} = h_i(C(n, f))$; see e.g., [BrHe, Section 5.2] for details.

Example 3.4 (The Alexander dual of a cyclic polytope). Let q, d be integers with $2 \leq q \leq d$. Put $n = 2d - q + 2$ and $f = 2(d - q + 1)$. Also, let $\Gamma = \Gamma_{n,f}$ be the boundary complex of a cyclic polytope $C(n, f)$, and let $\Delta = \Gamma^*$ be the Alexander dual of Γ . Then $k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution with $h = 1$.

Proof. Since Γ is a boundary complex of a simplicial f -polytope $C(n, f)$, $k[\Gamma]$ is a f -dimensional Gorenstein Stanley–Reisner ring with $a(k[\Gamma]) = 0$. Indeed, $[H_m^d(k[\Gamma])]_0 \cong \tilde{H}_{d-1}(\Gamma; k) \cong k \neq 0$. Also, we have $\text{indeg } k[\Gamma] = f/2 + 1 = d - q + 2 = n - d$ since $h_i(\Gamma) = \binom{n-f+i-1}{i}$ for all i . This implies that $k[\Gamma]$ has almost $(n - d)$ -linear resolution (see [Sch, TeHi]). On the other hand, we have

$$\dim k[\Delta] = n - \text{indeg } k[\Gamma] = d, \quad \text{indeg } k[\Delta] = n - \dim k[\Gamma] = n - f = q.$$

Thus the assertion easily follows from the above corollary and Theorem 3.2. \square

4. BUCHSBAUM STANLEY–REISNER RINGS WITH MINIMAL MULTIPLICITY

Let $A = k[A_1]$ be a homogeneous k -algebra of dimension d with the unique homogeneous maximal ideal $\mathfrak{m} = A_+$. In [Go2], Goto proved an inequality

$$e(A) \geq 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H_{\mathfrak{m}}^i(A))$$

and called the ring A a *Buchsbaum ring with minimal multiplicity* if equality holds. Also, he proved that a Buchsbaum homogeneous k -algebra with minimal multiplicity has maximal embedding dimension, and hence has 2-linear resolution.

In this section, in the class of Stanley–Reisner rings, we introduce the notion of *Buchsbaum ring with minimal multiplicity of type q* and prove that such a ring has a q -linear resolution; see Theorem 4.3. Furthermore, we give several characterizations of this notion. In the following, let c, d, q be integers with $c \geq 2$, $2 \leq q \leq d$.

Definition 4.1 (Minimal multiplicity of type q). Let $A := k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring with $\text{codim } A = c$ and $\text{indeg } A = q$. Then we say that A has *minimal multiplicity of type q* if

$$e(A) = \frac{c+d}{d} \binom{c+q-2}{q-2}.$$

Proposition 4.2. Let $A = k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring with $\text{indeg } A \geq 2$. Then the following conditions are equivalent:

- (1) A has minimal multiplicity in the sense of Goto [Go2].
- (2) A has minimal multiplicity of type 2.

(3) Δ is a finite disjoint union of $(d-1)$ -simplexes.

When this is the case, the normalization B of A is isomorphic to a finite product of polynomial rings with d -variables. In particular, the number of connected components of Δ is equal to the multiplicity of A .

Proof. (1) \implies (2) : Suppose that A has minimal multiplicity. Then since A has 2-linear resolution, $H_{\mathfrak{m}}^i(A) = 0$ for all $i \neq 1, d$ by Proposition 2.1. Thus $e := e(A) = 1 + h$ and $I(A) = (d-1)h$, where $h = \dim_k H_{\mathfrak{m}}^1(A)$. Also, since A has maximal embedding dimension, we have

$$n = \text{emb}(A) = e + d - 1 + I(A) = e + d - 1 + (d-1)(e-1) = de.$$

Hence $e = \frac{n}{d} = \frac{c+d}{d}$, as required.

(2) \iff (3) : Note that e is equal to the number of facets of Δ . Since each facet of Δ is a $(d-1)$ -simplex, we have $n \leq de$ by counting the vertices of Δ . Furthermore, equality holds if and only if Δ is a disjoint union of all facets.

(3) \implies (1) : Let $\Delta = \Delta_0 \cup \dots \cup \Delta_h$ be a simplicial complex such that each Δ_i is a $(d-1)$ -simplex. By Proposition 2.11, $A = k[\Delta]$ is a Buchsbaum ring of maximal embedding dimension. In particular, $H_{\mathfrak{m}}^i(A) = 0$ for all $i \neq 1, d$, and $h = \dim_k \tilde{H}_0(\Delta; k) = \dim_k H_{\mathfrak{m}}^1(A)$. Also, since Δ has $(1+h)$ facets, we get

$$e(A) = 1 + h = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(H_{\mathfrak{m}}^i(A)).$$

Hence A has minimal multiplicity. \square

The following theorem, which is a main theorem in this article, will justify our definition of minimal multiplicity of type q .

Theorem 4.3. *Let $A := k[\Delta]$ be a Buchsbaum Stanley–Reisner ring with $\text{codim } A = c$ and $\text{indeg } A = q$. Then*

(1) *The following inequality holds:*

$$(4.1) \quad e(A) \geq \frac{c+d}{d} \binom{c+q-2}{q-2}.$$

(2) *If A has minimal multiplicity of type q , then it has q -linear resolution.*

Proof. (1) Let $V = \{x_1, \dots, x_n\}$ be the vertex set of Δ where $n = c + d$. Put $\Gamma_i = \text{link}_{\Delta}(\{x_i\})$, and let \mathfrak{m}_i be the homogeneous maximal ideal of $k[\Gamma_i]$ for all i . Then $k[\Gamma_i]$ is a $(d-1)$ -dimensional Cohen–Macaulay ring since A is Buchsbaum. Also, we have that $\text{codim } k[\Gamma_i] = c$ and $\text{indeg } k[\Gamma_i] \geq q-1$ by the assumption.

By Lemma 2.5 (or see [EiGo, Corollary 1.11]), we get

$$e(k[\Gamma_i]) \geq \binom{c+(q-1)-1}{(q-1)-1} = \binom{c+q-2}{q-2}.$$

On the other hand, counting the number of facets of Δ , we have

$$d \cdot e(A) = \sum_{i=1}^n e(k[\Gamma_i]) \geq (c+d) \binom{c+q-2}{q-2},$$

as required.

(2) Suppose that the equality holds. Then $e(k[\Gamma_i]) = \binom{c+q-2}{q-2}$ for all i . It follows from Proposition 2.4 that $k[\Gamma_i]$ has $(q-1)$ -linear resolution. This implies that

$a(k[\Gamma_i]) = q - 1 - (d - 1) - 1 = q - d - 1$. Also, we have that $\text{indeg } A = q$ since $\text{indeg } A \geq q$ and $\text{indeg } k[\Gamma_i] = q - 2$. Thus the assertion follows from Theorem 4.4 below. \square

Theorem 4.4. *Let $A = k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring of Δ on $V = \{x_1, \dots, x_n\}$. Put $\text{indeg } A = q$.*

- (1) **(Improved version of Hibi’s criterion)** *A has q -linear resolution if and only if the following conditions are satisfied:*
 - (a) $\tilde{H}_{q-1}(\Delta; k) = 0$.
 - (b) $a(k[\text{link}_\Delta \{x_i\}]) \leq q - d$ for all $i = 1, \dots, n$.
- (2) *If $a(k[\text{link}_\Delta \{x_i\}]) = q - d - 1$ for all i , then A has q -linear resolution and $a(A) = q - d - 2$.*

Proof. Put $\Gamma_i = \text{link}_\Delta(\{x_i\})$ for all $i = 1, \dots, n$. Since $A = k[\Delta]$ is Buchsbaum, $k[\Gamma_i]$ is Cohen–Macaulay for all i and $H_{\mathfrak{m}}^p(A) = [H_{\mathfrak{m}}^p(A)]_0 = \tilde{H}_{p-1}(\Delta; k)$ for all $p \leq d - 1$. Also, by Hochster’s formula, we have

$$\begin{aligned} F(H_{\mathfrak{m}}^d(A), t) &= \sum_{F \in \Delta} \dim_k \tilde{H}_{d-\#(F)-1}(\text{link}_\Delta F; k) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{\#(F)}; \\ F(H_{\mathfrak{m}_i}^{d-1}(k[\Gamma_i]), t) &= \sum_{G \in \Gamma_i} \dim_k \tilde{H}_{d-\#(G)-2}(\text{link}_{\Gamma_i} G; k) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{\#(G)}. \end{aligned}$$

First we compute the a -invariant of A .

Claim 1:

- (1) (a), (b) $\implies [H_{\mathfrak{m}}^d(A)]_j = 0$ for all $j = -1, \dots, q - d$.
- (2) $\implies [H_{\mathfrak{m}}^d(A)]_j = 0$ for all $j = -1, \dots, q - d, q - d - 1$.

Suppose (a), (b) in (1). We may assume that $q \leq d - 1$. Then since $a(k[\Gamma_i]) \leq q - d \leq -1$, we have $[H_{\mathfrak{m}_i}^{d-1}(k[\Gamma_i])]_0 = [H_{\mathfrak{m}_i}^{d-1}(k[\Gamma_i])]_{q-d+1} = 0$ for all $i = 1, \dots, n$, where \mathfrak{m}_i denotes the homogeneous maximal ideal of $k[\Gamma_i]$.

Now let F be a face of Δ with $1 \leq \#(F) \leq d - q$. As F contains a vertex of Δ (say x_i), if we put $G = F \setminus \{x_i\}$, then $G \in \Gamma_i$ and $\text{link}_{\Gamma_i} G = \text{link}_\Delta F$. If $G \neq \emptyset$, then $1 \leq \#(G) = \#(F) - 1 \leq d - q - 1$. Then

$$\tilde{H}_{d-\#(F)-1}(\text{link}_\Delta F; k) = \tilde{H}_{d-\#(G)-2}(\text{link}_{\Gamma_i} G; k) = 0$$

because $[H_{\mathfrak{m}_i}^{d-1}(k[\Gamma_i])]_{q-d+1} = 0$. If $G = \emptyset$, then $F = \{x_i\}$. Thus

$$\tilde{H}_{d-\#(F)-1}(\text{link}_\Delta F; k) = \tilde{H}_{d-2}(\Gamma_i; k) = [H_{\mathfrak{m}_i}^{d-1}(k[\Gamma_i])]_0 = 0.$$

Hence $\tilde{H}_{d-\#(F)-1}(\text{link}_\Delta F; k) = 0$ for all $F \in \Delta$ with $1 \leq \#(F) \leq d - q$. This yields that $[H_{\mathfrak{m}}^d(k[\Delta])]_j = 0$ by Hochster’s formula.

Suppose (2). Then since $a(k[\Gamma_i]) = q - d - 1 \leq -1$, one can also prove the claim in this case by the similar argument as above. //

Claim 2: $[H_{\mathfrak{m}}^d(A)]_0 \cong \tilde{H}_{d-1}(\Delta; k) = 0$.

First suppose (a), (b) in (1). If $q = d$, then the assertion is clear by the assumption. So we may assume that $q \leq d - 1$. Let K_A be the graded canonical module of A , that is, $[K_A]_j = \text{Hom}_k([H_{\mathfrak{m}}^d(A)]_{-j}, k)$. Then $[K_A]_1 = 0$ by Claim 1. Thus

$$[K_A]_0 \subseteq \bigcap_{i=1}^n (0) :_{K_A} x_i = \text{Hom}_A(A/\mathfrak{m}, K_A) = 0,$$

where the last vanishing follows from $\text{depth} K_A > 0$. Thus $[H_m^d(A)]_0 = 0$, as required. In the case of (2), one can also prove the claim by the same argument as above. //

By virtue of the above two claims, we get

$$(4.2) \quad (1) \ a(A) \leq q - d - 1; \quad (2) \ a(A) \leq q - d - 2.$$

In particular, in the case of (2), we have $\text{reg } A \leq a(A) + d + 1 \leq q - 1$ by Hoa–Miyazaki theorem. On the other hand, $\text{reg } A \geq \text{indeg } A - 1 = q - 1$. Therefore A has q -linear resolution and $a(A) = q - d - 2$.

In order to prove that A has q -linear resolution in the case of (1), it suffices to show that $H_m^p(A) = 0$ for all $p = q, q + 1, \dots, d - 1$; see Theorem 1.7. Note that $H_m^q(A) \cong \tilde{H}_{q-1}(\Delta; k) = 0$ by the assumption. Also, we have that $\text{reg } A \leq q$ by Eq.(4.2). Hence $H_m^p(A) = [H_m^p(A)]_0 = 0$ for all $p = q + 1, \dots, d - 1$, as required. The converse follows from Hibi’s criterion. \square

Theorem 4.5 (Characterization of Buchsbaum complex with “minimal multiplicity of type q ”). *Let $A := k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring such that $\text{codim } A = c$ and $\text{indeg } A = q$. Let Δ^* denote the Alexander dual of Δ and put*

$$h_{c,d,q} = \frac{(c + q - 2) \cdots (c + 1)c}{d(d - 1) \cdots (d - q + 2)}.$$

Then the following conditions are equivalent:

- (1) *A has minimal multiplicity of type q , that is,*

$$e(A) = \frac{c + d}{d} \binom{c + q - 2}{q - 2}.$$

- (2) *A has q -linear resolution and $\dim_k \tilde{H}_{q-2}(\Delta; k) = h_{c,d,q}$.*

- (3) *The h -vector of A is*

$$\left(1, c, \dots, \binom{c + q - 2}{q - 1}, -\binom{d}{q} h, \binom{d}{q + 1} h, \dots, (-1)^{d - q + 1} \binom{d}{d} h \right),$$

where $h = h_{c,d,q}$.

- (4) *$k[\text{link}_\Delta \{x_i\}]$ has $(q - 1)$ -linear resolution for all i .*

- (5) *$a(k[\text{link}_\Delta \{x_i\}]) = q - d - 1$ for all i .*

- (6) *$a(A) = q - d - 2$.*

- (7) *$k[\Delta^*]$ is Cohen–Macaulay with pure and almost linear resolution and with $a(k[\Delta^*]) = 0$, that is, the graded minimal free resolution of $k[\Delta^*]$ over $S = k[x_1, \dots, x_n]$ ($n = c + d$) can be written as follows:*

$$0 \rightarrow S(-(c + d))^{\beta_q^*} \rightarrow S(-(c + q - 2))^{\beta_{q-1}^*} \rightarrow \cdots \rightarrow S(-c)^{\beta_1^*} \rightarrow S \rightarrow k[\Delta^*] \rightarrow 0.$$

When this is the case, $\beta_q^ = h_{c,d,q}$.*

Proof. It suffices to show the following implications: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$, $(1) \Rightarrow (5)$, $(4) \Leftrightarrow (5) \Leftrightarrow (6)$, and $(5), (6) \Rightarrow (7) \Rightarrow (2)$.

We first show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. If we suppose (1), then A has q -linear resolution by Theorem 4.3. Putting $h = \dim_k \tilde{H}_{q-2}(\Delta; k)$, we obtain that

by Proposition 2.4,

$$\frac{c+d}{d} \binom{c+q-2}{q-2} = e(A) = \binom{c+q-2}{q-2} - h \binom{d-1}{q-1}.$$

This implies that $h = h_{c,d,q}$. In particular, we get (2). Also, (2) \Rightarrow (3) follows from Theorem 2.6. If we suppose (3), then we have

$$\begin{aligned} e(A) &= \sum_{i=0}^d h_i = \sum_{i=0}^{q-1} \binom{c+i-1}{i} + (-1)^{q+1} \sum_{i=q}^d (-1)^i \binom{d}{i} h_{c,d,q} \\ &= \binom{c+q-1}{q} - \binom{d-1}{q-1} h_{c,d,q} = \frac{c+d}{d} \binom{c+q-2}{q-2}. \end{aligned}$$

Hence we get (1).

Next we show that (1) \Rightarrow (5), (4) \Leftrightarrow (5) \Leftrightarrow (6). (1) \Rightarrow (5) follows from the proof of Theorem 4.3. Since $k[\Gamma_i]$ is Cohen–Macaulay, (4) and (5) are equivalent. (5) \Rightarrow (6) follows from Theorem 4.4, and one can prove the converse similarly.

To complete the proof, we must show that (5), (6) \Rightarrow (7) \Rightarrow (2). Suppose that (5) and (6). By Theorem 4.4, A has q -linear resolution. Thus by Theorem 3.2, the Alexander dual $k[\Delta^*]$ is a Cohen–Macaulay homogeneous k -algebra with almost linear resolution. Namely, the graded minimal free resolution of $k[\Delta^*]$ over a polynomial ring S can be written as follows:

$$0 \rightarrow F_q \rightarrow F_{q-1} = S(-(c+q-2))^{\beta_{q-1}^*} \rightarrow \cdots \rightarrow F_1 = S(-c)^{\beta_1^*} \rightarrow S \rightarrow A^* \rightarrow 0,$$

where $F_q = S(-(c+d))^{\beta^*} \oplus S(-(c+q-1))^{\beta^{*'}}$ and $\beta^* = \dim_k H_{\mathfrak{m}}^{q-1}(A)$, $\beta^{*'} = \dim_k H_{\mathfrak{m}}^d(A)_{q-d-1}$. Since $a(A) = q-d-2$ by (6), we have $\beta^{*'} = 0$, that is, the above resolution is pure and thus $a(k[\Delta^*]) = 0$. Hence we get (7). Conversely, suppose (7). By Theorem 3.2, $A = k[\Delta]$ has q -linear resolution. On the other hand, since $k[\Delta^*]$ is a Cohen–Macaulay homogeneous k -algebra with pure resolution of type $(c_1, \dots, c_q) = (c, c+1, \dots, c+q-2, c+d)$, we have

$$\beta_q^* = (-1)^{q+1} \prod_{j=1}^{q-1} \frac{c_j}{c_j - c_q} = h_{c,d,q}.$$

by Herzog–Kühl’s formula (see [HeKu] or [BrHe, Theorem 4.1.15]). Combining with $h = \beta_q^*$, we obtain that $h = h_{c,d,q}$, as required. \square

In the following, we give some examples of Buchsbaum Stanley–Reisner rings with minimal multiplicity of type 3. We say that a simplicial complex Δ is spanned by a set S if S is the set of facets of Δ .

Example 4.6 (Hibi [Hi1]). Let $d \geq 2$ be an integer, and let k be a field. Put $n = 2d - 1$ and $V = \{1, 2, \dots, n\}$. Let Δ be the simplicial complex which is spanned by $S = \{\{\bar{i}, \bar{i}+1, \dots, \bar{i}+d-1\} \mid i = 1, 2, \dots, 2d-1\}$, where \bar{p} stands for $p \in V$ with $p \equiv q \pmod{2d-1}$.

Then $k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with minimal multiplicity of type 3.

Proof. Put $A = k[\Delta]$ and $c = n - d = d - 1$. First note that $\{i, j\} \in \Delta$ for any i, j with $1 \leq i < j \leq n$. On the other hand, $\{1, 2, d\} \notin \Delta$. Thus A is a d -dimensional equidimensional Stanley–Reisner ring with $\text{indeg } A = 3$. Also, we have

$$\text{link}_{\Delta}\{1\} = \{2, \dots, d\} \cup \{2d-1, 2, \dots, d-1\} \cup \cdots \cup \{d+1, d+2, \dots, 2d-1\}.$$

Hence $\text{link}_\Delta\{1\}$ is a Cohen–Macaulay complex which is a $(d-2)$ -tree (e.g. cf. [Te3]) and thus it has 2-linear resolution ([Fr2]). Similarly, $\text{link}_\Delta\{i\}$ is Cohen–Macaulay with 2-linear resolution. By Theorem 4.5(4), $k[\Delta]$ is a Buchsbaum ring with minimal multiplicity of type 3. \square

Example 4.7 (The Alexander dual of a cyclic polytope). Let q, d be integers with $2 \leq q \leq d$. Put $n = 2d - q + 2$ and $f = 2(d - q + 1)$. Let $C(n, f)$ be a cyclic polytope with n vertices. Also, let Δ be the Alexander dual of the boundary complex of $C(n, f)$. Then $k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with minimal multiplicity of type q .

In particular, Conjecture 2.9 is true for $h_{c,d,q} = 1$.

Proof. It follows from Example 3.4 and Theorem 4.5(2). \square

Example 4.8 ([Te1, Theorem 3.3]). Let n be an integer such that $n > 3$, and suppose that $2n + 1$ is a prime number. Let Δ be the simplicial complex on $V = \{1, 2, \dots, n\}$ which is spanned by

$$S = \{\{a, b, a + b\} \mid 1 \leq a < b, a + b \leq n\} \\ \cup \{\{a, b, c\} \mid 1 \leq a < b < c \leq n, a + b + c = 2n + 1\}.$$

Then $A = k[\Delta]$ is a 3-dimensional Buchsbaum Stanley–Reisner ring such that $e(A) = \frac{n(n-2)}{3}$. In particular, A has minimal multiplicity of type 3, and thus has 3-linear resolution.

Proof. By [Te1], A is Buchsbaum with $e(A) = \frac{n(n-2)}{3}$. If one put $c = n - 3$ and $d = q = 3$, then $e(A) = \frac{c+d}{d} \binom{c+q-2}{q-2}$. Thus A has minimal multiplicity of type 3 by definition. \square

In the above example, the assumption “ $2n + 1$ is prime” is superfluous in some sense. Indeed, if $n \equiv 0$ or $\equiv 2 \pmod{3}$, then there exists a 3-dimensional Buchsbaum Stanley–Reisner ring $k[\Delta]$ with minimal multiplicity which satisfies $\text{emb}(k[\Delta]) = n$; see Example 4.10 for details.

In general, $h_{c,d,q}$ is not an integer. For example, if $d = q = 3$, then since $h_{c,d,q} = \frac{c(c+1)}{6}$. However, Hanano constructed Buchsbaum complexes which satisfies $h = \dim_k H_m^2(A) = \lfloor h_{c,3,3} \rfloor$. Motivated by his work, we introduce the following notion.

Definition 4.9 (Maximal homology). Let $A = k[\Delta]$ be a d -dimensional Buchsbaum Stanley–Reisner ring with q -linear resolution. Put $c = \text{codim } A$ and $h = \dim_k H_m^{q-1}(A)$. We say that A has *maximal homology* if $h = \lfloor h_{c,d,q} \rfloor$ holds, where

$$h_{c,d,q} = \frac{(c+q-2) \cdots (c+1)c}{d(d-1) \cdots (d-q+2)}.$$

According to Theorem 4.5, A has minimal multiplicity of type q if and only if A has q -linear resolution of maximal homology and $h_{c,d,q}$ is an integer.

Example 4.10 (Hanano [Ha]). Let n be an integer with $n \geq 5$. Let Δ be the simplicial complex on V which is spanned by the following set S . Then $k[\Delta]$ is a 3-dimensional Buchsbaum Stanley–Reisner ring with 3-linear resolution of maximal homology. Furthermore, $k[\Delta]$ has minimal multiplicity of type 3 if and only if $n \equiv 0$ or $\equiv 2 \pmod{3}$.

(1) The case of $n \not\equiv 1 \pmod{3}$. Put $V := \{0, 1, \dots, n-1\}$ and

$$S = \left\{ \{\bar{i}, \overline{i+k}, \overline{i+2k}\} \mid 0 \leq i \leq k-1 \right\} \\ \cup \left\{ \{\bar{i}, \overline{i+k}, \overline{i+j}\} \mid 0 \leq i \leq 3k-1, k+1 \leq j \leq 2k-1 \right\},$$

when $n = 3k$ and

$$S = \left\{ \{\bar{i}, \overline{i+1}, \overline{i+3j+2}\} \mid 0 \leq i \leq 3k+1, 0 \leq j \leq k-1 \right\},$$

where $n = 3k+2$. Here \bar{p} stands for $q \in V$ with $p \equiv q \pmod{n}$.

(2) The case of $n \equiv 1 \pmod{3}$. Put $V := \{\infty, 0, 1, \dots, n-2\}$ and

$$S = \left\{ \{\infty, \bar{i}, \overline{i+1}\} \mid 0 \leq i \leq 3k-1 \right\} \\ \cup \left\{ \{\bar{i}, \overline{i+1}, \overline{i+3j}\} \mid 0 \leq i \leq 3k-1, 1 \leq j \leq k-1 \right\},$$

when $n = 3k+1$. Here \bar{p} stands for $q \in V$ with $p \equiv q \pmod{n-1}$.

Proof. (1) By [Ha], $k[\Delta]$ is a Buchsbaum ring with $\text{indeg } k[\Delta] = 3$ and

$$(1) \ h(\Delta) = \left(1, n-3, \frac{(n-2)(n-3)}{2}, -\frac{(n-2)(n-3)}{6} \right); \\ (2) \ h(\Delta) = \left(1, n-3, \frac{(n-2)(n-3)}{2}, -\frac{(n-1)(n-4)}{6} \right).$$

In particular,

$$e(k[\Delta]) = \begin{cases} \frac{n(n-2)}{3} & \text{in (1);} \\ \frac{(n-1)^2}{3} & \text{in (2).} \end{cases}$$

Therefore the assertion follows from the lemma below. \square

Lemma 4.11. *Let $A = k[\Delta]$ be a 3-dimensional Buchsbaum Stanley–Reisner ring of Δ on $V = \{x_1, \dots, x_n\}$ with $\text{indeg } k[\Delta] = 3$. Then*

- (1) $k[\Delta]$ has minimal multiplicity of type 3 if and only if $e(k[\Delta]) = \frac{n(n-2)}{3}$.
- (2) $k[\Delta]$ has maximal homology which is not minimal multiplicity of type 3 if and only if $e(k[\Delta]) = \frac{(n-1)^2}{3}$.

Proof. Since (1) just follows from the definition, it suffices to consider (2) only.

Suppose that $e(k[\Delta]) = \frac{(n-1)^2}{3}$. Then if we prove that $A = k[\Delta]$ has 3-linear resolution, then by Proposition 2.4 we have

$$\dim_k H_{\mathfrak{m}}^2(A) = \binom{n-1}{2} - e(A) = \frac{(n-1)(n-4)}{6} = [h_{e,3,3}].$$

Hence A has maximal homology. Thus it is enough to show that A has 3-linear resolution.

Put $\Gamma_i = \text{link}_{\Delta}\{x_i\}$ for all $i = 1, \dots, n$. By the similar argument as in the proof of Theorem 4.3, we have

$$(n-1)^2 = 3 \cdot e(A) = \sum_{i=1}^n e(k[\Gamma_i]) \geq n(n-2).$$

Note that $e(k[\Gamma_i]) \geq n-2$ and the equality holds if and only if $k[\Gamma_i]$ has 2-linear resolution. So we may assume that

$$e(k[\Gamma_1]) = n-1, \quad e(k[\Gamma_i]) = n-2 \text{ for all } i \geq 2.$$

Then $a(k[\Gamma_i]) = -1$ and thus $x_i[K_A]_0 = 0$ for all $i = 2, \dots, n$. Thus

$$[K_A]_0 \subseteq \bigcap_{i=2}^n (0) :_{K_A} x_i = \text{Hom}_A(A/(x_2, \dots, x_n)A, K_A) = 0,$$

where the last vanishing follows from $(x_2, \dots, x_n)A \notin \text{Ass}_A K_A = \text{Assh}(A)$. Hence $\tilde{H}_2(\Delta; k) \cong [H_m^3(A)]_0 = 0$. Thus A has 3-linear resolution, as required. \square

Remark 4.12. *There are two different Buchsbaum complexes with minimal multiplicity which has the same numerical data. For instance, if $n \geq 8$, $n \equiv 2 \pmod{3}$ and $2n+1$ is prime, then Δ in Example 4.8 is different from Δ in Example 4.10 since they have distinct links.*

The next example gives Buchsbaum Stanley–Reisner rings with minimal multiplicity of higher type.

Example 4.13 (Bruns–Hibi [BrHi, Proposition 3.2]). Let $n \geq 6$ be an even integer, and let k be a field. Let Γ be the simplicial complex whose facets are

$$\{\bar{i}, \overline{i+1}, \overline{i+2}\}, \{\bar{i}, \overline{i+1}, \overline{i+4}\}, \dots, \{\bar{i}, \overline{i+1}, \overline{i+(n-2)}\}, \quad i = 1, \dots, n,$$

where \bar{p} stands for $q \in [n]$ with $p \equiv q \pmod{n}$. Also, let Δ be the Alexander dual of Γ . Then $A = k[\Delta]$ is a $(n-3)$ -dimensional Buchsbaum Stanley–Reisner ring with minimal multiplicity of type $(n-3)$. Also, $\text{codim } A = 3$ and $\text{emb}(A) = n$.

Proof. $k[\Gamma]$ is a 3-dimensional Cohen–Macaulay Stanley–Reisner ring with pure and almost 3-linear resolution by [BrHi, Proposition 3.2]. Thus the required assertion follows from Theorem 4.5(7). \square

5. COHEN–MACAULAY COVER OF A BUCHSBAUM COMPLEX

In this section, we introduce the notion of *Cohen–Macaulay cover*, and prove that such a cover always exists for any Buchsbaum simplicial complex Δ with d -linear resolution ($d = \dim k[\Delta]$); see Theorem 5.2.

Also, we prove that if a complex Δ is between two Buchsbaum complexes with d -linear resolution on the same vertex set V , it is also d -linear Buchsbaum on V . Using these facts, we can reduce our problem to construct “Buchsbaum simplicial complexes with maximal homology” with given parameters. As an application, in the case of $d = 3$, we will prove that Conjecture 2.9 is true; see Theorem 5.9.

In the following, let Δ be a $(d-1)$ -dimensional simplicial complex on $V = \{x_1, \dots, x_n\}$ over a field k with $\text{indeg } k[\Delta] = d$.

Definition 5.1 (Cohen–Macaulay cover). A simplicial complex $\tilde{\Delta}$ on V is said to be a *Cohen–Macaulay (d -linear) cover* of Δ over k if $\tilde{\Delta}$ satisfies the following conditions:

- (1) $\tilde{\Delta}$ is a $(d-1)$ -dimensional complex which contains Δ as a subcomplex.
- (2) $k[\tilde{\Delta}]$ is a Cohen–Macaulay ring with d -linear resolution.

Theorem 5.2 (Existence of Cohen–Macaulay cover). *If $k[\Delta]$ is a d -dimensional Buchsbaum Stanley–Reisner ring with d -linear resolution, then there exists a Cohen–Macaulay (d -linear) cover $\tilde{\Delta}$ of Δ .*

Proof. We may assume that k is infinite. We use the following notation:

- $S = k[x_1, \dots, x_n]$, $A = k[\Delta] = S/I_\Delta$
- $h = \dim_k H_{\mathfrak{m}}^{d-1}(A) = \dim_k \widetilde{H}_{d-2}(\Delta; k)$, $c = n - d$.
- $e_{\text{CM}} = \binom{n-1}{d-1}$, $\mu_{\text{CM}} = \binom{n-1}{d}$.
- $\mathbb{F} = \{\{x_{i_1}, \dots, x_{i_d}\} \subseteq V \mid \{x_{i_1}, \dots, x_{i_d}\} \text{ is a facet of } \Delta\}$.
- $\mathbb{G} = \{G \subseteq V \mid \#(G) = d, G \in \mathbb{F}\}$.

Then we note that

$$e := e(A) = \#\mathbb{F} = e_{\text{CM}} - h, \quad \mu := \mu(I_\Delta) = \#\mathbb{G} = \mu_{\text{CM}} + h$$

by Proposition 2.4.

Take a homogeneous ideal $J \subseteq S$ such that JA is a minimal reduction of $\mathfrak{m}A = (x_1, \dots, x_n)A$. If necessary, we may assume that J is generated by the following elements f_{c+1}, \dots, f_n :

$$f_i = x_i - \sum_{j=1}^c c_{i,j} x_j \quad (c_{i,j} \in k, i = c+1, \dots, n).$$

First, we show the following claim:

Claim: $I_\Delta + J = (x_1, \dots, x_c)^d + J$ in S .

Actually, since A is Buchsbaum and JA is a parameter ideal of A , we have

$$l_S(S/I_\Delta + J) = l_A(A/JA) = e(JA) + I(A) = e(A) + h = e_{\text{CM}}.$$

Also, since A has d -linear resolution, by Theorem 1.7, we have

$$(x_1, \dots, x_c)^d + J \subseteq I_\Delta + J.$$

On the other hand, since

$$l_S(S/(x_1, \dots, x_c)^d + J) = \binom{c+d-1}{d-1} = e_{\text{CM}} = l_S(S/I_\Delta + J),$$

we obtain the required equality, and the claim is proved.

Now consider any term order $<$ on S , and put

$$\{M_1, \dots, M_\mu\} = \left\{ x_{i_1} \cdots x_{i_d} \mid \{x_{i_1}, \dots, x_{i_d}\} \in \mathbb{G} \right\},$$

where $M_1 < \dots < M_\mu$. Also, we put

$$\{N_1, \dots, N_{\mu-h}\} = \left\{ x_1^{k_1} \cdots x_c^{k_c} \mid k_1 + \dots + k_c = d, k_i \geq 0 \right\}.$$

Let \widetilde{M}_i be the homogeneous polynomial of degree d given by substituting $x_i = \sum_{j=1}^c c_{ij} x_j$ for M_i . Then \widetilde{M}_i can be written as a linear combination of $N_1, \dots, N_{\mu-h}$:

$$\widetilde{M}_i = a_{i1} N_1 + \dots + a_{i, \mu-h} N_{\mu-h}$$

for all $i = 1, \dots, \mu$. Since $I_\Delta + J$ generates $(x_1, \dots, x_c)^d$ in $S/J \cong k[x_1, \dots, x_c]$, the coefficient matrix $\mathbb{A} = (a_{ij}) \in \text{Mat}(k; \mu \times (\mu - h))$ satisfies that $\text{rank } \mathbb{A} = \mu - h$. In particular, there exist $(\mu - h)$ -distinct row vectors of \mathbb{A} which are linearly independent over k . Let G_1, \dots, G_h be elements of \mathbb{G} corresponding the other rows of \mathbb{A} , and put $\widetilde{\Delta} = \Delta \cup \{G_1, \dots, G_h\}$. Then $\widetilde{\Delta}$ be a $(d-1)$ -dimensional simplicial complex on V contains Δ . Also, if we put $\widetilde{A} = k[\widetilde{\Delta}]$, then

$$l_{\widetilde{A}}(\widetilde{A}/J\widetilde{A}) = \dim_k k[x_1, \dots, x_c]/(x_1, \dots, x_c)^d = e_{\text{CM}}$$

and $e(\tilde{A}) = e(A) + h = e_{\text{CM}}$. Hence \tilde{A} is Cohen–Macaulay. Furthermore, by Lemma 2.5, \tilde{A} has d -linear resolution, and $\tilde{\Delta}$ becomes a Cohen–Macaulay cover of Δ . \square

Let Δ^{\min} be a Buchsbaum simplicial complex on V with d -linear resolution and $\dim_k H_{\mathfrak{m}}^{d-1}(k[\Delta^{\min}]) = \lfloor h_{c,d,d} \rfloor = \lfloor \frac{(c+d-2) \cdots (c+1)c}{d!} \rfloor$.

Corollary 5.3. *If such a complex Δ^{\min} exists, then there exists a Cohen–Macaulay cover $\tilde{\Delta}$ of Δ^{\min} .*

In the following, we want to prove that a complex between two Buchsbaum complexes with d -linear resolution is also Buchsbaum with d -linear resolution. If this is true, it enables us to construct many Buchsbaum complexes with d -linear resolution as an application of Cohen–Macaulay cover. However, it is troublesome to prove the above claim by an induction because the links of a simplicial complex with d -linear resolution do not necessarily have linear resolution; see also Theorem 4.5. In order to get rid of this difficulty, we introduce the notion of d -fullness as follows.

Definition 5.4. Let $d \geq 2$ be an integer. Let Δ be a $(d-1)$ -dimensional simplicial complex on V . Then Δ is called d -full if Δ is pure and it contains all $(d-2)$ -faces of 2^V .

Remark 5.5. *Let k be any field. Δ is d -full if and only if $k[\Delta]$ is a d -dimensional equidimensional Stanley–Reisner ring with $\text{indeg } k[\Delta] \geq d$.*

If $k[\Delta]$ is a Buchsbaum Stanley–Reisner ring with d -linear resolution, then Δ is d -full.

Lemma 5.6. *Let $\Delta^- \subseteq \Delta \subseteq \Delta^+$ be simplicial complexes on V . Then the following statements hold.*

- (1) *If both Δ^- and Δ^+ are Cohen–Macaulay d -full complexes, then so is Δ .*
- (2) *If both Δ^- and Δ^+ are Buchsbaum d -full complexes, then so is Δ .*

Proof. Since Δ^- , Δ^+ are d -full, so is Δ . Indeed, as $\Delta^- \subseteq \Delta \subseteq \Delta^+$, we have that $d-1 = \dim \Delta^- \leq \dim \Delta \leq \dim \Delta^+ = d-1$. Thus $\dim \Delta = d-1$. In particular, since Δ can be written as $\Delta = \Delta^- \cup \{F_1, \dots, F_h\}$ for some facets of Δ^+ . Moreover, $\text{indeg } k[\Delta] \geq \text{indeg } k[\Delta^-] \geq d$.

(1) We want to prove that Δ is Cohen–Macaulay by an induction on $d = \dim k[\Delta] \geq 2$. First suppose that $d = 2$. Then since Δ^- is connected and Δ is a complex on the same vertex set V , Δ is also connected. That is, Δ is Cohen–Macaulay.

Next suppose that $d \geq 3$. Put $\Gamma_i = \text{link}_{\Delta}\{x_i\}$, $\Gamma_i^+ = \text{link}_{\Delta^+}\{x_i\}$, $\Gamma_i^- = \text{link}_{\Delta^-}\{x_i\}$ for all $i = 1, \dots, n$. Then Γ_i^+ and Γ_i^- are Cohen–Macaulay $(d-1)$ -full complexes. Applying the induction hypothesis to $\Gamma_i^- \subseteq \Gamma_i \subseteq \Gamma_i^+$, we obtain that each Γ_i is Cohen–Macaulay. Then Δ is Buchsbaum because Δ is pure and Γ_i is Cohen–Macaulay for all i . In particular, $H_{\mathfrak{m}}^{d-1}(k[\Delta]) \cong \tilde{H}_{d-2}(\Delta; k)$ and $H_{\mathfrak{m}}^p(k[\Delta]) \cong \tilde{H}_{p-1}(\Delta; k) = 0$ for all $p \leq d-2$.

Now consider the following diagram:

$$\begin{array}{ccccc}
C_{d-1}(\Delta^-) & \xrightarrow{\partial_{d-1}^-} & C_{d-2}(\Delta^-) & \xrightarrow{\partial_{d-2}^-} & C_{d-3}(\Delta^-) \\
\tau \downarrow & & id \downarrow & & id \downarrow \\
C_{d-1}(\Delta) & \xrightarrow{\partial_{d-1}} & C_{d-2}(\Delta) & \xrightarrow{\partial_{d-2}} & C_{d-3}(\Delta),
\end{array}$$

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where τ is injective and the last two vertical maps are identity maps. Since $\partial_{d-2}^- = \partial_{d-2}$ and τ is injective, we have that

$$0 = \tilde{H}_{d-2}(\Delta^-; k) = \frac{\text{Ker } \partial_{d-2}^-}{\text{Im } \partial_{d-1}^-} \longrightarrow \frac{\text{Ker } \partial_{d-2}}{\text{Im } \partial_{d-1}} = \tilde{H}_{d-2}(\Delta; k)$$

is surjective. This yields that $\tilde{H}_{d-2}(\Delta; k) = 0$ and thus $k[\Delta]$ is Cohen–Macaulay, as required.

(2) Considering the links of each vertex x_i , we have

$$\text{link}_{\Delta^-}\{x_i\} \subseteq \text{link}_{\Delta}\{x_i\} \subseteq \text{link}_{\Delta^+}\{x_i\}.$$

By the assumption, we have that two links of both sides are Cohen–Macaulay $(d-1)$ -full complexes on $V \setminus \{x_i\}$. By (1), $\text{link}_{\Delta}\{x_i\}$ is also Cohen–Macaulay. Hence Δ is Buchsbaum. \square

Theorem 5.7. *Let $\Delta^- \subseteq \Delta \subseteq \Delta^+$ be simplicial complexes on V . If both $k[\Delta^-]$ and $k[\Delta^+]$ are Buchsbaum Stanley–Reisner rings with d -linear resolutions, then so is $k[\Delta]$.*

Proof. Since $k[\Delta^+]$ (resp., $k[\Delta^-]$) is a Buchsbaum ring with d -linear resolution, Δ^+ (resp., Δ^-) is a Buchsbaum d -full complex. Thus Δ is Buchsbaum by the above lemma. Hence by Hibi’s criterion, it is enough to show $\tilde{H}_{d-1}(\Delta; k) = 0$.

Now consider the following diagram:

$$\begin{array}{ccccc} C_{d-1}(\Delta) & \xrightarrow{\partial_{d-1}} & C_{d-2}(\Delta) & \xrightarrow{\partial_{d-2}} & C_{d-3}(\Delta) \\ \tau \downarrow & & id \downarrow & & id \downarrow \\ C_{d-1}(\Delta^+) & \xrightarrow{\partial_{d-1}^+} & C_{d-2}(\Delta^+) & \xrightarrow{\partial_{d-2}^+} & C_{d-3}(\Delta^+), \end{array}$$

where τ is injective. Then

$$\tilde{H}_{d-1}(\Delta; k) = \text{Ker } (\partial_{d-1}) \hookrightarrow \text{Ker } (\partial_{d-1}^+) = \tilde{H}_{d-1}(\Delta^+; k) = 0$$

since $k[\Delta^+]$ has d -linear resolution. Hence $\tilde{H}_{d-1}(\Delta; k) = 0$, as required. \square

Example 5.8 (Real projective plane [Hi2, (5.2),(5.4)]). Let Δ be the simplicial complex on the vertex set $V = [6]$ whose maximal faces are $\{1, 2, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 4\}$, $\{1, 3, 6\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 6\}$, $\{3, 5, 6\}$ and $\{4, 5, 6\}$.

If $\text{char } k \neq 2$, then $k[\Delta]$ is a Cohen–Macaulay ring with 3-linear resolution. On the other hand, if $\text{char } k = 2$, then $k[\Delta]$ is a non-Cohen–Macaulay Buchsbaum ring with $\text{indeg } k[\Delta] = 3$, but it does not have linear resolution.

Put $\Delta' = \Delta \setminus \{4, 5, 6\}$ (Möbius band). Then $k[\Delta']$ is a Buchsbaum ring with 3-linear resolution and $h := \dim_k H_m^2(k[\Delta']) = 1$ in any characteristic. If $\text{char } k \neq 2$, then Δ is a Cohen–Macaulay cover of Δ' . On the other hand, if $\text{char } k = 2$, then $\Delta' \cup \{1, 4, 6\}$ is a Cohen–Macaulay cover of Δ' , but Δ is not.

Also, since $0 < h < 1 = h_{3,3,3} = 2$, $k[\Delta']$ does not have minimal multiplicity of type 3. However, one can easily see that Δ' cannot contain any Buchsbaum complex Δ'' having minimal multiplicity of type 3.

In the following, as an application of the notion of Cohen–Macaulay cover, we prove Conjecture 2.9 in the case of $d = q = 3$.

Theorem 5.9. *Let c, h be integers with $c \geq 1$. Then the following conditions are equivalent:*

- (1) *There exists a 3-dimensional Buchsbaum Stanley–Reisner ring $A = k[\Delta]$ with 3-linear resolution such that $\text{codim } A = c$ and $\dim H_{\mathfrak{m}}^2(A) = h$.*
- (2) *The following inequality holds:*

$$0 \leq h \leq h_{c,3,3} = \frac{(c+1)c}{6}.$$

Proof. It is enough to show that if $0 \leq h \leq \frac{(c+1)c}{6}$ then there exists a 3-dimensional Buchsbaum Stanley–Reisner ring $k[\Delta]$ with 3-linear resolution and $\dim H_{\mathfrak{m}}^2(k[\Delta]) = h$.

For any positive integer c , we have an example of Buchsbaum complex $\Delta^- := \Delta^{\min}$ with 3-linear resolution of maximal homology over k ; see Example 4.10 due to Hanano. That is, $\dim_k H_{\mathfrak{m}}^2(k[\Delta^-]) = \lfloor \frac{(c+1)c}{6} \rfloor =: h_0$. By Theorem 5.2, we can take a Cohen–Macaulay cover $\Delta^+ := \widetilde{\Delta^-}$ of Δ^- . Then $e(k[\Delta^+]) - e(k[\Delta^-]) = h_0$. For a given h , let Δ be a simplicial subcomplex of Δ^+ containing Δ^- with $(h_0 - h) + e(k[\Delta^-])$ facets. Then $k[\Delta]$ is a Buchsbaum Stanley–Reisner ring with 3-linear resolution and $\dim_k H_{\mathfrak{m}}^2(k[\Delta]) = h$ by Theorem 5.7, as required. \square

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DEPARTMENT OF MATHEMATICS, FACULTY OF CULTURE AND EDUCATION, SAGA UNIVERSITY,
SAGA 840–8502, JAPAN

E-mail address: `terai@cc.saga-u.ac.jp`

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA 464–8602, JAPAN

E-mail address: `yoshida@math.nagoya-u.ac.jp`